On the parametrization of the controllability subspaces of a controllable pair

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Abstract

Given a controllable linear control system defined by a pair of constant matrices (A, B), the set of controllability subspaces is an stratified submanifold of the set of (A, B)-invariant subspaces. We parameterize each strata by means of coordinate charts. This parametrization has significant differences to that of (A, B) invariant subspaces, showing a more complex geometric structure.

Introduction

Consider a time-invariant, linear multivariable system

$$\dot{x} = Ax + Bu$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $m \leq n$. If F is a state feedback and G is a nonsigular matrix, the controllable subspace of $(A + BF, BG)$ is called a *controllability subspace* of the original pair (A, B), Controllability subspaces play an important role in geometric control theory (significant references are [5], [9] and [10]). In [6] the geometry of the set of controllability subspaces of a given dimension has been studied. More precisely it is shown that the set of controllability subspaces $\mathcal{S}$ of a given dimension d, $\mathrm{Ctr}\ _d(A, B)$, can be stratified according to the controllability indices h of the restriction of (A, B) to $\mathcal{S}$. As shown in [6], the controllability subspaces are precisely those subspaces for which the restriction is controllable(see section 1). So, we have a finite partition

$$\mathrm{Ctr}\ _d(A, B) = \bigcup_h \mathrm{Ctr}\ _h(A, B)$$

where each $\mathrm{Ctr}\ _h(A, B)$ is an orbit space with a structure similar to that of $\mathrm{Inv}\ _h(B^t, A^t)$ (see [3] and [6]). However, since the restriction defining $\mathrm{Ctr}\ _h(A, B)$ is not the dual to that defined in a natural way by (B^t, A^t) (see [3]), the geometry of $\mathrm{Ctr}\ _h(A, B)$ and that of $\mathrm{Inv}\ _h(B^t, A^t)$ have significant differences. In particular, the coordinate atlas obtained in [7] can not be "translated" to the set of controlability subspaces. Our aim in this paper is to obtain a coordinate atlas

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parameterizing each one of the strata Ctr $_h(A, B)$. We point out that, in contrast with [4], where the structure of linked and non linked parameters shows that Inv $_h(B^t, A^t)$ is a vector bundle on a flag manifold (see also [8]), in Ctr $_h(A, B)$ the situation is much more involved.

In this paper we make use of the following notation. $\mathbb{K}$ is the field of either the complex or real numbers. $\mathcal{M}_{p,q}$ denotes the set of $p \times q$ matrices with entries in $\mathbb{K}$ and $\mathcal{M}^*_{p,q}$ the set of full rank ones. If $p = q$ we write simply $\mathcal{M}_p$ and $\mathcal{M}^*_p$, respectively. If $X \in \mathcal{M}_{p,q}$ we identify X with the linear map $\mathbb{K}^q \longrightarrow \mathbb{K}^p$ defined in a natural way.

1 Preliminaries

We fix a controllable pair (A, B) with $A \in \mathcal{M}_n$ and $B \in \mathcal{M}_{n,m}$ and controllability indices $\underline{k} = (k_1 \geq \cdots \geq k_r)$. We will assume without lost of generality that B has full column rank m.

We recall that a subspace $\mathcal{S}$ of $\mathbb{K}^n$ is an (A, B)-invariant subspace if $A(\mathcal{S}) \subset \mathcal{S} + \text{Im } B$. The subspace $\mathcal{S}$ is said to be a *controllability subspace* of (A, B) if there exists $F \in \mathcal{M}_{n,m}$ and $G \in \mathcal{M}_{m,\ell}$ such that

$$\mathcal{S} = \text{Im } BG + \text{Im } (A + BF)G + \cdots + \text{Im } (A + BF)^{n-1}BG\,.$$

It is clear that a controllability subspace of (A, B) is an (A, B)-invariant subspace.

A characterization of controllability subspaces in terms of a restriction on (A, B)-invariant subspaces is given in [6]. We recall now the definition of this restriction in an equivalent formulation.

Let $\mathcal{S}$ be an (A, B)-invariant subspace and let $F \in \mathcal{M}_{m,n}$ such that $(A + BF)\mathcal{S} \subset \mathcal{S}$. Let $s = \dim(\mathcal{S} \cap \text{Im } B)$ and $\mathcal{S} \cap \text{Im } B = \text{Im } (BG)$ with G an $m \times s$ full rank matrix. If $\mathcal{S} = \text{Im } X$ where X is a $n \times d$ full rank matrix we have from the above relations that $(A + BF)X = X\overline{A}$ and $BG = X\overline{B}$ where $\overline{A} \in \mathcal{M}_d$ and $\overline{B} \in \mathcal{M}_d$ are uniquely determined by these equalities.

Lemma 1.1 *The pair $(\overline{A}, \overline{B})$ is well defined modulo feedback equivalence.*

Proof. Let $F' \in \mathcal{M}_{m,n}$, $P \in \mathcal{M}^*_d$, $Q \in \mathcal{M}^*_s$ and $\overline{A}'$, $\overline{B}'$ be such that $(A + BF')XP = XP\overline{A}'$, $BGQ = XP\overline{B}'$. We have to show that $(\overline{A}', \overline{B}')$ is feedback equivalent to $(\overline{A}, \overline{B})$. If we keep the matrix F and change X and G by XP and GQ, respectively, our statement follows easily. So, we can suppose that $P = I_d$, $Q = I_s$. Then we can write $(A + BF')X = X\overline{A}'$ as

$$(A + BF)X + BHX = X\overline{A}'$$

with $H = F' - F$. But, $(A + BF)X = X\overline{A}$. Hence

$$X(\overline{A}' - \overline{A}) = BHX\,.$$

So, Im $(BHX) \subset \mathcal{S} \cap \text{Im } B = \text{Im } BG$, and we can define a linear map $\overline{F} : \mathbb{R}^d \longrightarrow \mathbb{R}^s$ such that $BHX = BG\overline{F}$ (recall that BG is full rank). Then

$$X(\overline{A}' - \overline{A}) = BG\overline{F} = X\overline{BF}$$

and the lemma follows. ■

Definition 1.2 *With the above notation we define $(\overline{A}, \overline{B})$ a* restriction *of (A, B) to $\mathcal{S}$. It is well defined modulo feedback equivalence.*

Remark 1.3 *One can check that the relations defining* $(\overline{A},\overline{B})$ *are equivalent to the existence of matrices* $Y \in \mathcal{M}_{m,d}$ *and* $G \in \mathcal{M}_{m,s}$ *making commutative the following diagram*

$$\begin{array}{ccc} \mathbb{K}^d \times \mathbb{K}^s & \xrightarrow{(\overline{A},\overline{B})} & \mathbb{K}^d \\ {\scriptstyle \left(\begin{smallmatrix} X & 0 \\ Y & G \end{smallmatrix}\right)}\big\downarrow & & \big\downarrow{\scriptstyle X} \\ \mathbb{K}^n \times \mathbb{K}^m & \xrightarrow[(A,B)]{} & \mathbb{K}^n \end{array}$$

where $s = \dim(\mathcal{S} \cap \operatorname{Im} B)$ *and the vertical arrows are full rank matrices (we can always put* $Y = FX$ *for a suitable* $F : \mathbb{K}^m \longrightarrow \mathbb{K}^n$*). Then,*

$$\operatorname{Im} \begin{pmatrix} X & 0 \\ Y & G \end{pmatrix} = \{(x,y);\, x \in \mathcal{S},\ Ax + By \in \mathcal{S}\}\ .$$

In fact, the inclusion $\subset$ *follows from the commutativity of the diagram. Conversely, let* $(x,y) \in \mathbb{K}^n \times \mathbb{K}^m$ *such that* $x \in \mathcal{S}$ *and* $Ax+By \in \mathcal{S}$*. Since* $x \in \mathcal{S}$ *we have that* $x = Xu$*. Let* $y = Yz+Gv$*. The commutativity of the diagram, which is equivalent to the equalities* $AX + BY = X\overline{A}$ *and* $BG = X\overline{B}$*, implies that* $BY(z-u) \in \mathcal{S}$*. But* $\mathcal{S} \cap \operatorname{Im} B = \operatorname{Im} BG$ *and* B *is injective. Therefore,* $Y(z-u) \in \operatorname{Im} G$ *and then* $z = u$ *((*$Y|G$*) has full rank), following our assertion.*

Remark 1.4 *Let* f, π *be the maps from* $\mathbb{K}^n \times \mathbb{R}^m$ *to* $\mathbb{K}^n$ *defined by* $f(x,y) = Ax + By$ *and* $\pi(x,y) = x$*, respectively. In [6] a more intrinsic definition of the above restriction is given in terms of the pair* (f,π)*. In fact, the equality proved in the preceding remark says that*

$$\operatorname{Im} \begin{pmatrix} X & 0 \\ Y & G \end{pmatrix} = \pi^{-1}(\mathcal{S}) \cap f^{-1}(\mathcal{S})$$

so that $(\overline{A},\overline{B})$ *is the matrix of the restriction of* (A,B) *to* $\pi^{-1}(\mathcal{S}) \cap f^{-1}(\mathcal{S}) \longrightarrow \mathcal{S}$ *in a suitable basis. This links the definition 1.2 with the definition of restriction given in [6], which generalizes the one given in [1].*

In [2] all the possible controllability indices of $(\overline{A},\overline{B})$ with regard to those of (A,B) are described. On the other hand, in [6] it is proved that an (A,B)-invariant subspace $\mathcal{S}$ is a controllability subspace if and only if $(\overline{A},\overline{B})$ is controllable. Moreover, if we denote by $\operatorname{Ctr}_{\,h}(A,B)$ the set of controllability subspaces $\mathcal{S}$ of (A,B) such that $\underline{h} = (h_1 \geq \cdots \geq h_s)$ are the controllability indices of a restriction $(\overline{A},\overline{B})$ of A,B to $\mathcal{S}$, $\operatorname{Ctr}_{\,h}(A,B)$ is described as an orbit space. Let us recall the main result. Let $s = \dim(\mathcal{S} \cap \operatorname{Im} B)$ and denote by $M(k,h)$ the set of matrices X such that

(a) $X \in \mathcal{M}^*_{n,d}$, $d = \dim S$.

(b) $X = [X_{ij}]$, $1 \leq i \leq r$, $1 \leq j \leq s$ with

$$X_{ij} = \begin{pmatrix} x^1_{i,j} & \dots & x^{h_j-k_i+1}_{i,j} & 0 & 0 & \dots & 0 \\ 0 & x^1_{i,j} & \dots & x^{h_j-k_i+1}_{i,j} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & x^1_{i,j} & \dots & x^{h_j-k_i+1}_{i,j} \end{pmatrix}$$

if $k_i \leq h_j$ or 0 otherwise.

Remark 1.5 *Notice that* $s = \dim(\mathcal{S} \cap \operatorname{Im} B)$ *is equivalent to* $\operatorname{rank}\,([X|B]) = d + m - s$*. Notice also that* $s = \operatorname{rank}\,\overline{B}$*.*

If $k = h$, we write $M(h,h) = G(h)$. Then, the following result is proved in [6]

Theorem 1.6 *With the above notation,*

(i) $G(h)$ *is a Lie subgroups of* $\mathrm{Gl}\,(d)$ *which acts freely on* $M(k,h)$ *on the right by matrix multiplication.*

(ii) The orbit space $M(k,h)/G(h)$ *has a differentiable structure such that the natural projection* $\pi : M(k,h) \longrightarrow M(k,h)/G(h)$ *is a submersion.*

(iii) The map $X \longmapsto \mathrm{Im}\, X$*, with* $X \in M(k,h)$ *induces a bijection between* $M(k,h)/G(h)$ *and* $Ctr_h(A,B)$*. Through this bijection* $Ctr_h(A,B)$ *is a differentiable manifold.*

(iv) $\dim Ctr_h(A,B) = \dim M(k,h) - \dim G(h) =$

$$= \sum_{1\le i\le r, 1\le j\le s} \sup\{k_j - k_i + 1, 0\} - \sum_{1\le i,j\le s} \sup\{h_j - h_i + 1, 0\} =$$

$$= \sum_{i=1}^{h} s_i((r_1 - s_1) - (r_{i+1} - s_{i+1}))$$

where $\underline{r} = (r_1 \ge \cdots \ge r_k)$*,* $\underline{s} = (s_1 \ge \cdots \ge s_h)$ *are the conjugate partitions of* $\underline{k}$ *and* $\underline{h}$*, respectively.*

Notice that $s_1 = \mathrm{rank}\ \overline{B} = \mathrm{rank}\ (BG) = \dim(S \cap \mathrm{Im}\ B) = s$.
If we reorder the Brunovsky basis we obtain a matrix representation of the subspaces in $Ctr_h(A,B)$ more convenient for our purposes. We illustrate it with an example.
Consider $k = (4,3,3,1,1)$ and $h = (3,3,1)$. Then, $S = \mathrm{Im}\ X$ where $X \in M(k,h)$ has the form

$$X = \left(\begin{array}{ccc|ccc|c}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
x_1 & 0 & 0 & x_9 & 0 & 0 & 0 \\
0 & x_1 & 0 & 0 & x_9 & 0 & 0 \\
0 & 0 & x_1 & 0 & 0 & x_9 & 0 \\
\hline
x_2 & 0 & 0 & x_{10} & 0 & 0 & 0 \\
0 & x_2 & 0 & 0 & x_{10} & 0 & 0 \\
0 & 0 & x_2 & 0 & 0 & x_{10} & 0 \\
\hline
x_3 & x_4 & x_5 & x_{11} & x_{12} & x_{13} & x_{17} \\
\hline
x_6 & x_7 & x_8 & x_{14} & x_{15} & x_{16} & x_{18}
\end{array}\right)$$

Denote by

$$(v_{11}, v_{12}, v_{13}, v_{14};\ v_{21}, v_{22}, v_{23};\ v_{31}, v_{32}, v_{33};\ v_{41}, v_{51})$$

and

$$(u_{11}, u_{12}, u_{13};\ u_{21}, u_{22}, u_{23};\ u_{31})$$

the corresponding bases of $\mathbb{K}^n$ and S, respectively.

If we arrange the above basis in the following way

$$\begin{array}{lcl} v_{11}, v_{12}, v_{13}, v_{14} & & v_{51}, v_{41}, v_{33}, v_{23}, v_{14} \\ v_{21}, v_{22}, v_{23} & & v_{32}, v_{22}, v_{13} \\ v_{31}, v_{32}, v_{33} & \longrightarrow & v_{31}, v_{21}, v_{12} \\ v_{41} & & v_{11} \\ v_{51} & & \end{array}$$

$$\begin{array}{lcl} u_{11}, u_{12}, u_{13} & & u_{31}, u_{23}, u_{13} \\ u_{21}, u_{22}, u_{23} & \longrightarrow & u_{22}, u_{12} \\ u_{31} & & u_{21}, u_{11} \end{array}$$

the matrix representation of S in these basis is

$$Z = \left[\begin{array}{ccc|cc|cc} x_{18} & x_{16} & x_8 & x_{15} & x_7 & x_{14} & x_6 \\ x_{17} & x_{13} & x_5 & x_{12} & x_4 & x_{11} & x_3 \\ 0 & x_{10} & x_2 & 0 & 0 & 0 & 0 \\ 0 & x_9 & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & x_{10} & x_2 & 0 & 0 \\ 0 & 0 & 0 & x_9 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & x_{10} & x_2 \\ 0 & 0 & 0 & 0 & 0 & x_9 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Remark 1.7 *Let P be the permutation matrix representing the above change of basis. Then,*

$$P^{-1}B = \begin{bmatrix} I_m \\ 0 \end{bmatrix},$$

so that rank $[X|B]) = m + d - s_1$ *if and only if* rank $Z_0 = d - s_1$*, Z_0 being the submatrix of Z obtained by removing the first r_1 rows and s_1 columns.*

Definition 1.8 *We denote by $\mathcal{M}(r, s)$ the set of matrices Z such that*

(α) $Z \in \mathcal{M}^*_{n,d}$, $d = \dim S$

(β) $Z = [Z_{ij}]$, $1 \leq i \leq k$, $1 \leq j \leq h$, *where $Z_{i,j}$ is a $r_i \times s_j$-matrix with*

- *(β_1)* $Z_{ij} = 0$ *if* $1 \leq j \leq h$, $j \leq i \leq k$
- *(β_2)* $Z_{ij} = [Z_{pq}^{j-i+1}]$, $i \leq p \leq k$, $j \leq q \leq h$ *with* Z_{pq}^{j-i+1} *of size* $(r_p - r_{p+1}) \times (s_q - s_{q+1})$ *and* $Z_{pq}^{j-i+1} = 0$ *if* $1 + i \leq p \leq k$, $p < q \leq k$.

(γ) rank $Z_0 = d - s_1$, *where* $Z_0 = [Z_{ij}]$, $2 \leq i \leq k$ *and* $2 \leq j \leq h$ *(see remark 1.7).*

If $r = s$ we write $\mathcal{M}(r, s) = \mathcal{G}(s)$

According to the block decomposition of Z in the above definition, the matrix Z of the above example corresponds to the partitions $\underline{r} = (5, 3, 3, 1)$ and $\underline{s} = (3, 2, 2)$, which are the conjugate partitions of k and h, respectively, and it is the block matrix

$$\begin{array}{cc}
\left[\begin{array}{ccc|cc|c}
Z^1_{11} & Z^1_{12} & Z^1_{13} & Z^2_{12} & Z^2_{13} & Z^3_{13} \\
0 & Z^1_{22} & Z^1_{23} & 0 & Z^2_{23} & 0 \\
0 & 0 & Z^1_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & Z^1_{22} & Z^1_{23} & Z^2_{23} \\
0 & 0 & 0 & 0 & Z^1_{33} & 0 \\
0 & 0 & 0 & 0 & 0 & \\
\hline
0 & 0 & 0 & 0 & 0 & Z^1_{33} \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
&
\begin{array}{c}
r_1-r_2 \\ r_2-r_3 \\ r_3-r_4 \\ r_4 \\ r_2-r_3 \\ r_3-r_4 \\ r_4 \\ r_3-r_4 \\ r_4 \\ r_4
\end{array}
\\
\begin{array}{cccccc}
s_1-s_2 & s_2-s_3 & s_3 & s_2-s_3 & s_3 & s_3
\end{array}
&
\end{array}$$

with

$$\begin{aligned}
&Z^1_{11} = \begin{bmatrix} x_{18} \\ x_{17} \end{bmatrix}, \; Z^1_{12} = \emptyset, \; Z^1_{13} = \begin{bmatrix} x_{16} & x_8 \\ x_{13} & x_5 \end{bmatrix}, \\
&Z^1_{22} = \emptyset, \; Z^1_{23} = \emptyset \\
&Z^1_{33} = \begin{bmatrix} x_{10} & x_2 \\ x_9 & x_1 \end{bmatrix} \\
&Z^2_{12} = \emptyset, \; Z^2_{13} = \begin{bmatrix} x_{15} & x_7 \\ x_{12} & x_4 \end{bmatrix} \\
&Z^2_{23} = \emptyset \\
&Z^3_{13} = \begin{bmatrix} x_{14} & x_6 \\ x_{11} & x_3 \end{bmatrix}
\end{aligned}$$

Remark 1.9 *The matrix Z of definition 1.7 can be derived easily from the following two rules*

(i) Each block $Z_{i+1,j+1}$ is obtained from Z_{ij} by removing the first $r_i - r_{i+1}$ rows and the first $s_j - s_{j+1}$ columns. Hence only different parameters can appear in the upper blocks $Z_{11}, Z_{12}, \ldots, Z_{1h}$.

(ii) Z, as well as each one of its Z_{ij} blocks, is an upper block triangular matrix.

Let $\underline{r} = (r_1 \geq \cdots \geq r_k)$ and $\underline{s} = (s_1 \geq \cdots \geq s_h)$ be conjugate partitions of k and h, respectively. Then, the natural map

$$M(k,h) \longrightarrow \mathcal{M}(r,s)$$

consisting on a change of basis by fixed permutation matrices is a diffomeomorphism inducing a bijection

$$M(k,h)/G(h) \cong \mathcal{M}(r,s)/\mathcal{G}(s).$$

Then, one can replace in theorem 1.6 $M(k,h)$ and $G(h)$ by $\mathcal{M}(r,s)$ and $\mathcal{G}(s)$, respectively.

As already said in [2] the compability conditions between the Brunovsky indices of a pair and its resctriction to an (A,B)-invariant subspaces in order to the set $\mathcal{M}(r,s)$ be not empty were described. These conditions are as follows (see [2, Corollary 3.3]):

$$(i) \qquad r_i \leq s_i + (r_1 - s - 1), \quad i = 1, \ldots, n \tag{1}$$

and

$$(ii) \qquad \sum_{j=1}^{h_p}(r_j - s_j - p) \geq 0, \quad 1 \leq p \leq r_1 - s_1, \tag{2}$$

where $h_p := \max\{i : r_i - s_i \geq p\}$, $p = 1, \ldots, r_1 - s_1$.
Notice that the inequality in (1) extends up to n. It may happen that $k_1 < h_1$ although $r_1 \geq s_1$ always. Thus we will assume that $s_i := 0$ for $i > h_1$ and $r_i := 0$ for $i > k_1$.

2 An atlas of coordinate charts of $Ctr_h(A, B)$

The manifold $Ctr_h(A, B)$ can be parameterized through a set of coordinate charts obtained as a system of canonical representatives of the orbits of its matrix description $\mathcal{M}(r, s)/\mathcal{G}(s)$. The algorithm for reducing an element of $\mathcal{M}(r, s)$ to a canonical form is based on a sequence of elementary transformations defined by some subsets of $\mathcal{G}(s)$. Let us write explicitly an element $P \in \mathcal{G}(s)$. $P = [P_{ij}]$ with

$$P_{ij} = \begin{bmatrix} P^{\alpha}_{ij} & P^{\alpha}_{i,j+1} & \cdots & \cdots & P^{\alpha}_{i,h} \\ 0 & P^{\alpha}_{i+1,j+1} & \cdots & \cdots & P^{\alpha}_{i+1,h} \\ 0 & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & P^{\alpha}_{i+h-j,h} \\ 0 & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

$1 \leq i, j \leq h$, $i \leq j$ and 0 otherwise ($\alpha = j - i + 1$).
From the action of P on $Z \in \mathcal{M}$ a canonical representative of the orbit $Z\mathcal{G}(s)$ can be derived. For convenience we introduce the following notation.

(i) If $Z = [Z_{ij}]$ and $Z_{ij} = [Z^{j-i+1}_{pq}]$ we write for $\ell = 1, \ldots, h$ and $q \geq \ell$

$$Z^{\ell}_q = \begin{bmatrix} Z^{\ell}_{1q} \\ Z^{\ell}_{2q} \\ \vdots \\ Z^{\ell}_{\delta_{q\ell},q} \end{bmatrix}$$

where $\delta_{q\ell} = \min(q - \ell + 1, k)$.

So,

$$Z_{1j} = \left[\begin{array}{c|c|c|c} Z^j_j & Z^j_{j+1} & \cdots & Z^j_h \\ \hline 0 & & & \\ & \overline{} & & \\ & 0 & & \\ & & & \overline{} \\ & & & 0 \end{array}\right] \qquad 1 \leq j \leq h$$

(ii) We denote by:

((ii)$_1$) $\prod_i$ a block diagonal matrix $P \in \mathcal{G}(s)$, such that

$$P_{11} = \text{diag}\,(I_{s_1-s_2}, I_{s_2-s_3}, \ldots, P^1_{ii}, \ldots, I_{s_h}), \quad 1 \leq i \leq h$$

(We recall that $\prod_i$ is quite determined).

((ii)$_2$) $\overline{\prod}_{ij}^{\alpha}$ a matrix $P \in \mathcal{G}(s)$ such that the only possible non zero block is P_{ij}^{α}, $\alpha \geq 2, 1 \leq i \leq j - \alpha + 1$.

((ii)$_3$) $\prod_{ij}^{\alpha} = I_d + \overline{\prod}_{ij}^{\alpha}$.

We call the matrices $\prod_i$ and $\prod_{ij}^{\alpha}$ *elementary matrices* and the corresponding actions, *elementary actions.*

The following proposition, whose proof is left to the reader, describes the effect on a matrix $Z \in \mathcal{M}(r, s)$ of these elementary actions. In fact we can limit ourselves to consider the action on the upper blocks $Z_{11}, \ldots, Z_{1h}$.

Proposition 2.1 *With the above notation the following holds*

1. *The upper blocks of $Z \prod_i$ are the same as those of Z except the blocks $Z_i^1, \ldots, Z_i^i$ which become $Z_i^1 P_{ii}^1, \ldots, Z_i^i P_{ii}^1$, respectively.*
2. *The upper blocks of $Z \prod_{ij}^{\alpha}$ are the same as those of Z except the blocks $Z_j^{\alpha}, \ldots, Z_j^{\alpha+i-1}$ which become $Z_j^{\alpha} + Z_i^1 P_{ij}^{\alpha}, \ldots, Z_j^{\alpha+i-1} + Z_i^i P_{ij}^{\alpha}$.*

Notice that the second action consists of adding to a block Z_j^{ℓ} linear combinations of the columns of the blocks $Z_1^1 \ldots, Z_{j-l+1}^1$.

We proceed now to describe the process of reduction for a matrix $Z \in \mathcal{M}(r, s)$.

Step 1. We begin with the block $Z_1^1 = Z_{11}^1$ of size $(r_1 - r_2) \times (s_1 - s_2)$. Since $s_1 - s_2 \leq r_1 - r_2$ because of the full rank condition of Z, we can choose $s_1 - s_2$ linearly independent rows, $n_{11} < n_{12} < \cdots < n_{1s_1-s_2}$. Then we take P_{11}^1 so that the submatrix of $Z_{11}^1 P_{11}^1$ formed by these rows is the identity matrix. Now, we can find matrices $\prod_{1j}^1$ making zeros the rows $n_{11}, \ldots, n_{1s_1-s_2}$ of the blocks Z_{1j}^1, $j = 2, \ldots, h$. Similarly, with matrices $\prod_{1j}^{\alpha}$ we make zero the same rows of all blocks Z_{1j}^{α}.

Step 2. We look at the submatrix of Z_2^1 obtained by removing the first $r_1 - r_2$ rows (see remark 1.7) and the rows $n_{11}, \ldots, n_{1s_1-s_2}$. This is actually the submatrix of the $(1, 1)$-block of Z_0 obtained by removing the rows $n_{11}, \ldots, n_{1s_1-s_2}$. Since Z_0 has full column rank, this submatrix has also full column rank $s_2 - s_3$. Thus we can choose $s_2 - s_3$ linearly independent rows $n_{21} < n_{22} < \cdots < n_{2s_2-s_3}$ with $n_{21} \geq r_1 - r_2$.

Then we take a matrix P_{22}^1 so that the submatrix of $Z_2^1 P_{22}^1$ formed by this second set of rows is the identity matrix. Then with matrices $\prod_{2j}^1$ we make zero the rows $n_{21}, \ldots, n_{2s_2-s_3}$ of Z_j^1, j=3,…,h, and with matrices $\prod_{2j}^{\alpha}$ we make zero the same rows of the blocks Z_j^{α}. Notice that the unit vector of the rows $n_{21}, \ldots, n_{2s_2-s_3}$ *we are not allowed* to make zero elements of the blocks $Z_{12}^2, Z_{13}^3, \ldots$

Step 3. We look at the submatrix of Z_3^1 obtained by removing the first $r_1 - r_2$ rows and the rows $n_{11}, \ldots, n_{1s_1-s_2}, n_{21}, \ldots, n_{2s_2-s_3}$ and we proceed in an analogous way as in the previous step. ■

The process ends after a finite number of steps and proves the following result.

Theorem 2.2 *For every $Z \in \mathcal{M}(r, s)$ there exists a set of positive integers pairwise different*

$$\begin{aligned} I = \{n_{ij};\ & 1 \leq n_{11} \leq \cdots \leq n_{1s_1-s_2} \leq r_1 - r_2\,, \\ & r_1 - r_2 \leq n_{i1} \leq \cdots \leq n_{is_i-s_{i+1}} \leq r_2 - r_{i+1}\,,\ i = 2, \ldots, h\} \end{aligned}$$

and a matrix $P \in \mathcal{G}(s)$ such that the matrix $Y = ZP$ satisfies the following conditions:

If $Y = [Y_{ij}]$*, with* $Y_{ij} \in \mathcal{M}_{r_i,s_j}$*, where* $Y_{ij} = 0$ *for* $1 \le j \le h$*,* $j < i \le k$ *and* $Y_{ij} = [L_{pq}^{j-i+1}]$*,* $i \le p \le k$*,* $j \le q \le h$*, with*

$$L_{pq}^{j-i+1} = 0\,, \quad i+1 \le p \le k, q < p \le k\,,$$

L_{pq}^{j-i+1} *of size* $(r_p - r_{p+1}) \times (s_q - s_{q+1})$*, then*

(i) For $q \ge 1$*, if*

$$L_q^1 = \begin{bmatrix} L_{1q}^1 \\ \vdots \\ \vdots \\ L_{qq}^1 \end{bmatrix}$$

the rows n_{ij} *with* $1 \le i \le q-1$*,* $1 \le j \le s_1 - s_q$ *are zero and the rows* $n_{q1}, \ldots, n_{qs_q - s_{q+1}}$ *are unit vectors.*

(ii) For $\alpha = 2, \ldots, h$ *and* $q \ge \alpha$*, if*

$$L_q^\alpha = \begin{bmatrix} L_{1q}^\alpha \\ \vdots \\ \vdots \\ L_{q-\alpha+1,q}^\ell \end{bmatrix}$$

the rows n_{ij} *with* $1 \le i \le q-1$*,* $1 \le j \le s_1 - s_q$ *are zero.*

(iii) The matrix $Y_0 = [Y_{ij}]$*,* $2 \le i \le k$*,* $2 \le j \le h$ *must have full rank, what is equivalent to* $det(Y_0^* Y_0) \neq 0$*.*

Definition 2.3 *We call the matrix* Y *a* reduced form *of* Z *and the set of indices* I *verifying the conditions given in theorem 2.2 an* admissible set of indices *for* Z*.*

Remark 2.4 *Notice that if* N *is the number of parameters in* Y*, we can decompose* N *as* $N = N_1 + N_2$*, where* N_1 *is the number of free parameters and* N_2 *is the number of parameters in* Y_0*. We call these last parameters, the* linked *parameters of* Y*.*

We illustrate this theorem with the following examples.

Examples 2.5 *Let* $Z \in \mathcal{M}((6,3,1),(4,2,1))$*. Taking the set of admissible indices* $n_{1,1} = 1, n_{1,2} = 3, n_{2,1} = 4, n_{3,1} = 5$*, the corresponding reduced form is*

$$Y = \left[\begin{array}{cccc|cc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1 & x_2 & x_3 & x_5 & x_7 & x_8 & x_{10} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & x_4 & 1 & 0 & x_9 & 0 \\
0 & 0 & 0 & x_6 & 0 & 0 & 0 \\
\hline
 & & & & 1 & 0 & 0 \\
 & & & & x_4 & 1 & x_9 \\
 & & & & 0 & x_6 & 0 \\
\hline
 & & & & & & x_6
\end{array}\right]$$

In this case, $N = 10$*, which coincides with* $\dim \mathrm{Ctr}_{(3,2,1,1)}(A, B)$ *according to the formula given in theorem 1.6. The controllability indices of* (A, B) *in this example are* $k = (3, 2, 1, 1, 1, 1)$*. Also, the matrix*

$$Y = \begin{bmatrix} 1 & 0 & 0 \\ x_4 & 1 & x_9 \\ 0 & x_6 & 0 \\ 0 & 0 & x_6 \end{bmatrix}$$

must have full rank, which is equivalent to $x_6 \neq 0$*. So,* $N_1 = 9, N_2 = 1$*.*

Examples 2.6 *If* $Z \in \mathcal{M}((6, 5, 4), (4, 3, 3, 2))$*, taking the set of integers admissible for* Z

$$\begin{aligned} &n_{11} = 1 \\ &n_{31} = 2 \\ &n_{41} = 3, \quad n_{42} = 4. \end{aligned}$$

The corresponding reduced form is as follows

$$Y = \left[\begin{array}{cccc|ccc|ccc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & x_1 & 0 & 0 & 0 & x_2 & x_3 & 0 & 0 \\
0 & x_4 & 1 & 0 & 0 & x_5 & x_6 & 0 & 0 & 0 & 0 & 0 \\
0 & x_7 & 0 & 1 & 0 & x_8 & x_9 & 0 & 0 & 0 & 0 & 0 \\
0 & x_{10} & x_{11} & x_{12} & 0 & x_{13} & x_{14} & 0 & 0 & 0 & 0 & 0 \\
0 & x_{15} & x_{16} & x_{17} & 0 & x_{18} & x_{19} & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & & & 1 & 0 & 0 & x_1 & 0 & 0 & x_2 & x_3 \\
 & & & & x_4 & 1 & 0 & 0 & x_5 & x_6 & 0 & 0 \\
 & & & & x_7 & 0 & 1 & 0 & x_8 & x_9 & 0 & 0 \\
 & & & & x_{10} & x_{11} & x_{12} & 0 & x_{13} & x_{14} & 0 & 0 \\
 & & & & x_{15} & x_{16} & x_{17} & 0 & x_{18} & x_{19} & 0 & 0 \\
\hline
 & & & & & & & x_4 & 1 & 0 & x_5 & x_6 \\
 & & & & & & & x_7 & 0 & 1 & x_8 & x_9 \\
 & & & & & & & x_{10} & x_{11} & x_{12} & x_{13} & x_{14} \\
 & & & & & & & x_{15} & x_{16} & x_{17} & x_{18} & x_{19}
\end{array}\right]$$

As in the previous example, the number of parameters in Y *is* 19 *which coincides with the dimension of* $\dim Ctr_{(4,4,3,1)}(A, B)$*. The controllability indices of* (A, B) *in this example are* $k = (3, 3, 3, 3, 2, 1, 1)$*. Also*

$$Y_0 = \left[\begin{array}{ccc|ccc|cc}
1 & 0 & 0 & x_1 & 0 & 0 & x_2 & x_3 \\
x_4 & 1 & 0 & 0 & x_5 & x_6 & 0 & 0 \\
x_7 & 0 & 1 & 0 & x_8 & x_9 & 0 & 0 \\
x_{10} & x_{11} & x_{12} & 0 & x_{13} & x_{14} & 0 & 0 \\
x_{15} & x_{16} & x_{17} & 0 & x_{18} & x_{19} & 0 & 0 \\
\hline
 & & & x_4 & 1 & 0 & x_5 & x_6 \\
 & & & x_7 & 0 & 1 & x_8 & x_9 \\
 & & & x_{10} & x_{11} & x_{12} & x_{13} & x_{14} \\
 & & & x_{15} & x_{16} & x_{17} & x_{18} & x_{19}
\end{array}\right]$$

must have full rank. Notice that in this case, there is no free parameters, that is to say, $N_1 = 0, N_2 = 19$*.*

In the previous examples we see that the number of parameters of the reduced forms coincide with the dimension of $\mathrm{Ctr}_h(A, B)$. In fact, this is a general result as we next show.

Proposition 2.7 *With the notation in the above theroem, if N is the number of parameters of Y, we have that*

$$N = \dim Ctr_h(A, B)\,.$$

Proof. According to the description of Y in theorem 2.2,

$$\begin{aligned}
N &= \sum_{i=1}^{h}(s_i - s_{i+1})(r_1 - s_1 - r_{i+1} + s_{i+1}) + \\
&\quad + \sum_{i=2}^{h}(s_i - s_{i+1})(r_1 - s_1 - r_i + s_i) + \cdots + s_h(r_1 - r_2 - s_1 + s_2) = \\
&= \sum_{i=1}^{h}(s_i - s_{i+1})(r_1 - s_1 - r_{i+1} + s_{i+1}) + \\
&\quad + \sum_{i=1}^{h}(s_{i+1} - s_{i+2})(r_1 - s_1 - r_{i+1} + s_{i+1}) + \cdots \\
&\quad + \sum_{i=1}^{h}(s_{i+h-1} - s_{i+h})(r_1 - s_1 - r_{i+1} + s_{i+1}) = \\
&= \sum_{1=1}^{h} s_i((r_1 - s_1) - (r_{i+1} - s_{i+1})) = \dim Ctr_h(A, B).
\end{aligned}$$

■

Our aim is to assign to each orbit $Z\mathcal{G}(s)$ a reduced form depending uniquely on an admissible set of indices. The next two lemmas show that this is possible.

Lemma 2.8 *Let $Z \in \mathcal{M}(r, s)$ and $Q \in \mathcal{G}(s)$. If I is an admissible set of indices for Z, it is also an admissible set of indices for ZQ.*

Proof. Let $Y = ZP$ be a reduced form for Z corresponding to an admissible set of indices $I = (n_{ij})$. Then $ZQ = Y(P^{-1}Q)$. So, we can assume without loss of generality that Z is in reduced form, and it is sufficient to look at the block Z_{11}. Then, if

$$\begin{aligned}
Z_{11} &= \begin{bmatrix} L^1_{11} & L^1_{12} & L^1_{13} & \dots \\ 0 & L^1_{22} & L^1_{23} & \dots \\ 0 & 0 & L^1_{33} & \dots \\ \dots & \dots & \dots & \end{bmatrix} \\
Z_{11}Q &= \begin{bmatrix} L^1_{11}Q^1_{11} & L^1_{11}Q^1_{12} + L^1_{12}Q^1_{22} & L^1_{11}Q^1_{13} + L^1_{12}Q^1_{23} + L^1_{13}Q^1_{33} & \dots \\ 0 & L^1_{22}Q^1_{22} & L^1_{22}Q^1_{23} + L^1_{23}Q^1_{33} & \dots \\ 0 & 0 & L^1_{33}Q^1_{33} & \dots \\ \dots & \dots & \dots & \end{bmatrix}
\end{aligned}$$

Since Q^1_{11} and Q^1_{22} are invertible, it is clear that n_{1j}, $1 \le j \le s_1 - s_2$ and n_{2j}, $1 \le j \le s_2 - s_3$ are admissible and we can assume that the corresponding rows in $L^1_{11}Q^1_{11}$ and $L^1_{22}Q^1_{22}$ are unit vectors, or what is equivalent, that $Q^1_{11} = I_{s_1 - s_2}$ and $Q^1_{22} = I_{s_2 - s_3}$.

Then with the block L^1_{22} we can make zero the block $L^1_{22}Q^1_{23}$ so that taking into account that Q^1_{33} is invertible we see that n_{3j}, $1 \le j \le s_3 - s_4$ is also admissible for $Z_{11}Q$. We reason in a similar way for the remainder sets of indices.■

Lemma 2.9 *Let Y and $\overline{Y}$ be two matrices of $\mathcal{M}(r,s)$ in reduced form with the same set of indices $I = (n_{ij})$. If $\overline{Y} = YP$ with $P \in \mathcal{G}(s)$, then $P = I_d$.*

Proof. The equality $\overline{Y} = YP$ implies that

$$\begin{aligned} &\overline{L}_{11}^1 = L_{11}^1 P_{11}^1 \\ &\overline{L}_{12}^1 = L_{11}^1 P_{12}^1 + L_{12}^1 P_{22}^1 \\ &\text{etc.} \end{aligned}$$

From these equalities and taking into account where the rows that are unit vectors or zero are placed, we conclude that $P = I_d$.■

We are now ready to parameterize the manifold $Ctr_h(A,B)$. More precisely we are going to describe a coordinate atlas of $Ctr_h(A,B)$. As we have seen, every point of $Ctr_h(A,B)$ can be identified with an orbit $Z\mathcal{G}(s)$ of $\mathcal{M}(r,s)$, so that taking into account the above lemmas we can associate to every point of $S \in Ctr_h(A,B)$ a matrix in reduced form Y depending only on a set of admissible set of indices $I = (n_{ij})$ (definition 2.3).

Furthermore, from the process that we used to obtain a reduced form we see that if $Z \in \mathcal{M}(r,s)$, there is an open neighbourhood of Z in $\mathcal{M}(r,s)$ such that for every matrix in this neighbourhood we can choose the same admissible set of indices I.

So, if we denote $\bigwedge$ the set of indices $I = (n_{ij})$ verifying the conditions in theorem 2.2 and $\mathcal{U}_I$ is the set of matrices $Z \in \mathcal{M}(r,s)$ such that I is admissible for Z, one has that $\{\mathcal{U}_I;\, I \in \bigwedge\}$ is an open covering of $\mathcal{M}(r,s)$. Hence, if $\pi : \mathcal{M}(r,s) \longrightarrow \mathcal{M}(r,s)(\mathcal{G}(s))$ is the natural projection, then $\{\pi(\mathcal{U}_I) = \widetilde{\mathcal{U}}_I;\, I \in \bigwedge\}$ is an open covering of $\mathcal{M}(r,s)/\mathcal{G}(s)$ and hence of $Ctr_h(A,B)$.

Finally, with the notation in remark 2.4, if we fix an order in the set of parameters of a reduced form, we can define the mapping

$$\varphi_I : \mathcal{U}_I \longrightarrow \mathbb{K}^{N_1} \times \mathcal{V}$$

in the following way: for every $Z \in \mathcal{U}_I$, $\varphi(Z)$ is the point in $\mathbb{K}^{N_1} \times \mathcal{V}$ defined by the N_1 free parameters of the reduced form Y of Z corresponding to I and the N_2 linked parameters of Y. Notice that, according to condition (iii) in theorem 2.2 and the compatibility conditions (1) and (2), $\mathcal{V}$ is an open and dense subset of $\mathbb{K}^{N_2}$. Taking into account the way we obtained Y, it turns out that φ_I is differentiable. The mapping φ_I induces a mapping

$$\theta_I : \widetilde{\mathcal{U}}_I \longrightarrow \mathbb{K}^{N_1} \times \mathcal{V}$$

and we can state the following result whose proof is as in [7]:

Theorem 2.10 *With the above notation θ_I is a diffeomorphism and $\{\widetilde{\mathcal{U}}_I,\, I \in \bigwedge\}$ is a coordinate atlas of $\mathcal{M}(r,s)/\mathcal{G}(s)$ and hence of $Ctr_h(A,B)$.*

3 Conclusions

Each one of the reduced forms described in theorem 2.2 (depending on the set of admissible indices) parameterizes an open and dense set of controllability subspaces of Ctr $_h(A,B)$, that is to say, "almost all" of them. The set Ctr $_h(A,B)$ is a subset of all the (A,B)-invariant subspaces (of dimension d) and one can obtain a parametrization of this set via the parametrization of (C,A)-invariant subspaces of dimension $n-d$ given, for example, in [7]. It is interesting to remark that, in contrast to this parametrization, the parametrization of Ctr $_h(A,B)$ obtained here has, in general, linked parameters, that is to say, we do not parameterize with $\mathbb{K}^N$, as in [7], but with the complementary of an algebraic variety, namely, $\mathbb{K}^{N_1} \times \mathcal{V}$. For example, in example 2.5 $\mathcal{V}$ is defined by $x_6 \neq 0$ and in example 2.6 $\mathcal{V}$ is defined by $\det Y_0^* Y_0 \neq 0$.

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